Stochastic Calculus for Fractional Brownian Motion. I: Theory[1]

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Abstract

This paper describes some of the results in [5] for a stochastic calculus for a fractional Brownian motion with the Hurst parameter in the interval $(1/2, 1)$. Two stochastic integrals are defined with explicit expressions for their first two moments. Multiple and iterated integrals of a fractional Browinian motion are defined and various properties of these integrals are given. A square integrable functional on a probability space of a fractional Brownian motion is expressed as an infinite series of multiple integrals.

1 Introduction

Fractional Brownian motion is a family of Gaussian processes that are indexed by the Hurst parameter H in the interval $(0, 1)$. These processes were introduced by Kolmogorov [10]. The first application of these processes was made by Hurst [7], [8] who used them to model the long term storage capacity of reservoirs along the Nile River. Mandelbrot [12] used these processes to model some economic time series and most recently these processes have been used to model telecommunication traffic (e.g., [11]). Two important properties of these Gaussian processes for modeling are self similarity and, for $H \in (1/2, 1)$, a long range dependence. The self similarity means that if $a > 0$ then $(B^H(at),\, t \geq 0)$ and $(a^H B^H(t),\, t \geq 0)$ have the same probability law where $(B^H(t),\, t \geq 0)$ is a (standard) fractional Brownian motion. The long range dependence means that if $r(n) = \mathbb{E}[B^H(1)(B^H(n+1) - B^H(n))]$ then $\sum_{n=1}^{\infty} r(n) = \infty$.

Now a fractional Brownian motion is defined. For each $H \in (0, 1)$, a real-valued Gaussian process $(B^H(t),\, t \geq 0)$ is defined such that $\mathbb{E}[B^H(t)] = 0$ and

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}]$$

for all $s, t \in \mathbb{R}_+$. If $H = 1/2$ then the fractional Brownian motion is a standard Brownian motion (Wiener process). These processes have a version with continuous sample paths. In this paper H is restricted to the interval $(1/2, 1)$. The pth variation of such a process is nonzero and finite for $p = 1/H$, that is, if $(P_n,\, n \in \mathbb{N})$ is sequence of partitions of $[0, 1]$ that are refinements of the previous and become dense in [0,1] then

$$\lim_{n\to\infty} \sum \left| B^H(t_i^{(n)}) - B^H(t_{i-1}^{(n)}) \right|^p = c(p) \qquad \text{a.s.}$$

where $P_n = \{t_0^{(n)}, \ldots, t_n^{(n)}\}$ and $c(p) = \mathbb{E}|B^H(1)|^p$ (e.g., [13]). For $H > 1/2$, $(B^H(t),\, t \geq 0)$ is not a semimartingale and not Markov. These facts require that a different stochastic calculus be used.

In this paper some results of a stochastic calculus from [5] are described. This description complements [4]. Some other approaches to stochastic calculus have been given in [1], [2], [3]. In Section 2, a directional derivative in the path space is given and two stochastic integrals with respect to a fractional Brownian motion are defined. The Wick product and the Hermite polynomials are introduced. In Section 3, multiple and iterated integrals with respect to a fractional Brownian motion are shown to satisfy many properties that are satisfied for the analogous integrals with respect to a Brownian motion. A square integrable functional on a probability space of a fractional Brownian motion is expressed as an infinite series of multiple integrals, which generalizes the well known result for Brownian motion.

2 Some Methods for Stochastic Calculus

Let $\Omega = C_0(\mathbb{R}_+, \mathbb{R})$ be the Fréchet space of real-valued continuous functions on $\mathbb{R}_+$ with initial value zero and the topology of local uniform convergence. There is a probability measure, P^H, on $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the Borel σ-algebra such that on the probability space $(\Omega, \mathcal{F}, P^H)$, the coordinate process is a fractional Brow-

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nian motion, that is,

$$B^H(t,\omega) = \omega(t)$$

for each $t \in \mathbb{R}_+$ and (almost all) $\omega \in \Omega$.

Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be given by

$$\phi(t) = H(2H-1)|t|^{2H-2}. \tag{1}$$

It follows directly that

$$\mathbb{E}[B^H(t)B^H(s)] = \int_0^t \int_0^s \phi(u-v)dudv. \tag{2}$$

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be Borel measurable. The function $f \in L^2_\phi$ if

$$|f|^2_\phi = \int_0^\infty \int_0^\infty f(s)f(t)\phi(s-t)dsdt < \infty \tag{3}$$

The Hilbert space L^2_ϕ is naturally associated with the Gaussian process $(B^H(t),\, t \geq 0)$. The inner product on L^2_ϕ is denoted by $\langle \cdot, \cdot \rangle_\phi$.

A notion of directional derivative in Ω in directions associated with L^2_ϕ is important in some computations with stochastic integrals.

Definition 2.1 *The ϕ-derivative of a random variable $F \in L^p$ in the direction Φg for $g \in L^2_\phi$ is defined as*

$$D_{\Phi g}F(\omega) = \lim_{\delta \to 0} \frac{1}{\delta}\left[F\left(\omega + \delta\int_0^{\cdot}(\Phi g)(s)ds\right) - F(\omega)\right]$$

if the limit exists in L^p where

$$(\Phi g)(t) = \int_0^\infty \phi(t-u)g(u)du$$

and $t \geq 0$. Furthermore, if there is a process $(D^\phi_s F,\, s \geq 0)$ such that

$$D_{\Phi g}F = \int_0^\infty D^\phi_s F g(s)ds$$

for each $g \in L^2_\phi$ then the random variable F is said to be ϕ-differentiable.

The notion of ϕ-differentiability is also defined for a process.

Definition 2.2 *The process $(F(t),\, t \geq 0)$ is said to be ϕ-differentiable if for each $t \in \mathbb{R}_+$, $F(t)$ is ϕ-differentiable and $D^\phi F : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is jointly measurable.*

The Wick product of two random variables is denoted $\diamond$. This product is important in the construction of the stochastic integrals (of Itô type).

Definition 2.3 *Let $\mathcal{L}(0,T)$ be the family of processes on $[0,T]$ such that $F \in \mathcal{L}(0,T)$ if $\mathbb{E}|F|^2_\phi < \infty$, F is ϕ-differentiable, the trace of $(D^\phi_s F_t,\, s,t \in [0,T])$ exists and $\mathbb{E}\int_0^T (D^\phi_s F_s)^2 ds < \infty$ and for each sequence of partitions $(\pi_n,\, n \in \mathbb{N})$ of $[0,T]$ such that $|\pi_n| \to 0$ as $n \to \infty$*

$$\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} |D^\phi_s F_{t_i^{(n)}} - D^\phi_s F_s| ds\right]^2$$

and

$$\mathbb{E}|F^\pi - F|^2_\phi$$

tend to 0 as $n \to \infty$ where $\pi_n = \{t_0^{(n)}, \ldots, t_n^{(n)}\}$ and F^π is the simple process induced by π_n.

The stochastic integral of $F \in \mathcal{L}(0,T)$ is constructed from Riemann sums using the Wick product as

$$\sum_{i=0}^{n-1} F^\pi_{t_i} \diamond (B^H(t_{i+1}) - B^H(t_i)). \tag{4}$$

Theorem 2.1 *Let F be a process in $\mathcal{L}(0,T)$. The limit in $L^2(P)$ of Riemann sums of the form (4) exists for each sequence of partitions $(\pi_n,\, n \in \mathbb{N})$ such that $|\pi_n| \to 0$ as $n \to \infty$ and the limit does not depend on the sequence of partitions. This limit is denoted as $\int_0^T F dB^H$. Furthermore, $\mathbb{E}[\int_0^T F dB^H] = 0$ and*

$$\mathbb{E}\left|\int_0^T F dB^H\right|^2 = \mathbb{E}\left[\left(\int_0^T D^\phi_s F_s ds\right)^2 + |F|^2_\phi\right]. \tag{5}$$

A stochastic integral of Stratonovich type is now defined.

Definition 2.4 *Let $(\pi_n,\, n \in \mathbb{N})$ be a sequence of partitions of $[0,T]$ such that $|\pi_n| \to 0$ as $n \to \infty$ and is dense. If the sequence of random variables*

$$\left(\sum_{i=0}^{n-1} F(t_i^{(n)})(B^H(t_{i+1}^{(n)}) - B^H(t_i^{(n)}))\right)$$

converges in $L^2(P)$ to the same limit for each sequence of partitions, then this limit is called the stochastic integral of Stratonovich type and the limit is denoted $\int_0^T F\delta B^H$.

The two stochastic integrals are related in the following result.

Theorem 2.2 *If $F \in \mathcal{L}(0,T)$, then the stochastic integral of Stratonovich type exists and the following equality is satisfied*

$$\int_0^T F\delta B^H = \int_0^T F dB^H + \int_0^T D^\phi_s F_s ds \quad a.s. \tag{6}$$

The sequence of Hermite polynomials $(H_n,\, n \in \mathbb{N})$ where $\deg H_n = n$ can be defined by a generating function as

$$e^{tx-(1/2)t^2} = \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!}.$$

Define

$$\tilde{f}(t) = |f1_{[0,t]}|_\phi^{-1} \int_0^t f dB^H$$

and

$$H_n^{\phi,f}(t) = |f1_{[0,t]}|_\phi^n H^n(\tilde{f}(t)).$$

As an application of an Itô formula for fractional Brownian motion (Theorem 4.3, [5]) there is the following result.

Proposition 2.1 *If $f1_{[0,T]} \in L_\phi^2$, then the following equality is satisfied*

$$dH_n^{\phi,f}(t) = nH_{n-1}^{\phi,f}(t)f(t)dB^H(t)$$

3 Multiple Integrals

Let $f \in L_\phi^2$ be such that $|f|_\phi = 1$. The Wick exponential, $\exp^\diamond$, and the Wick logarithm, $\log^\diamond$, are defined as

$$\exp^\diamond\left(\int_0^\infty f dB^H\right) := \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_0^\infty f dB^H\right)^{\diamond n}$$

and

$$\log^\diamond\left(1+\int_0^\infty f dB^H\right) := \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n}\left(\int_0^\infty f dB^H\right)^{\diamond n}$$

where $(\int_0^\infty f dB^H)^{\diamond n}$ is the nth Wick power of $\int_0^\infty f dB^H$. This nth Wick power can be expressed in terms of the Hermite polynomial H_n.

Lemma 3.1 *If $f \in L_\phi^2$ with $|f|_\phi = 1$ then*

$$\left(\int_0^\infty f dB^H\right)^{\diamond n} = H_n(\int_0^\infty f dB^H)^{\diamond n}$$

for each $n \in \mathbb{N}$ where H_n is the Hermite polynomial of degree n.

More generally, if $f \in L_\phi^2$ then

$$\begin{aligned}\left(\int_0^\infty f dB^H\right)^{\diamond n} &= |f|_\phi^n\left(\frac{\int_0^\infty f dB^H}{|f|_\phi}\right)^{\diamond n}\\ &= |f|_\phi^n H_n\left(\frac{\int_0^\infty f dB^H}{|f|_\phi}\right).\end{aligned}$$

The Wick exponential can be expressed in terms of the usual exponential as follows.

Proposition 3.1 *If $f \in L_\phi^2$, then*

$$\exp^\diamond\left(\int_0^\infty f dB^H\right) = \exp\left(\int_0^\infty f dB^H - \frac{1}{2}|f|_\phi^2\right). \quad (7)$$

This exponential (7) is the Radon-Nikodym derivative of the following translate of a fractional Brownian motion

$$X(t) = B^H(t) + \int_0^t (\Phi f)(s)ds$$

and

$$(\Phi f)(t) = \int_0^\infty \phi(t,u)f(u)du.$$

The following expectation is useful in computations with multiple integrals of a fractional Brownian motion.

Lemma 3.2 *If $f_1, \ldots, f_n, g_1, \ldots, g_m \in L_\phi^2$, then the following equality is satisfied*

$$\begin{aligned}&\mathbb{E}\Bigg[\left(\int_0^\infty f_1 dB^H \diamond \cdots \diamond \int_0^\infty f_n dB^H\right)\\ &\quad\times\left(\int_0^\infty g_1 dB^H \diamond \cdots \diamond \int_0^\infty g_m dB^H\right)\Bigg]\\ &= \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{n!}\sum_\sigma \langle f_1, g_{\sigma(1)}\rangle_\phi \cdots \langle f_n, g_{\sigma(n)}\rangle_\phi & \text{if } m = n\end{cases}\end{aligned}$$

where $\sum_\sigma$ denotes the sum over all permutations σ of $\{1,\ldots,n\}$.

The Hilbert space L_ϕ^2 is extended to its nth symmetric tensor product, that is,

$$L_{\phi,n}^2 := L_\phi^2 \otimes \cdots \otimes L_\phi^2.$$

If $f \in L_{\phi,n}^2$, that is, $f: \mathbb{R}_+^n \to \mathbb{R}$ and is symmetric in its arguments then

$$\begin{aligned}\langle f, f\rangle_\phi := &\int_{\mathbb{R}_+^n} \phi(u_1-v_1)\cdots\phi(u_n-v_n)f(u_1,\ldots,u_n)\\ &\times f(v_1,\ldots,v_n)du_1\cdots du_n dv_1\cdots dv_n.\end{aligned}$$

If $f \in L_{\phi,n}^2$ is of the form

$$f(s_1,\ldots,s_n) = \sum a_{k_1\cdots k_n} e_{k_1}(s_1)\cdots e_{k_n}(s_n)$$

and $(e_n,\, n \in \mathbb{N})$ is a complete orthonormal basis of L_ϕ^2, then the multiple integral of f, $I_n(f)$ is defined as

$$I_n(f) = \sum a_{k_1\cdots k_n}\int_0^\infty e_{k_1} dB^H \cdots \int_0^\infty e_{k_n} dB^H. \quad (8)$$

This definition of multiple integral is easily extended to an arbitrary $f \in L_{\phi,n}^2$.

The following result gives the expectation of a product of two multiple integrals.

Lemma 3.3 *If $f \in L^2_{\phi,n}$ and $g \in L^2_{\phi,m}$, then*

$$\mathbb{E}[I_n(f)I_m(f)] = \begin{cases} \langle f, g\rangle_\phi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

The iterated integral can be defined by the natural recursion

$$\int_{0\le s_1<\cdots<s_n\le t} f(s_1,\dots,s_n)dB^H(s_1)\cdots dB^H(s_n)$$
$$= \int_0^t \left(\int_{0\le s_1<\cdots\le s_n} f(s_1,\dots,s_n) \times dB^H(s_1)\cdots dB^H(s_{n-1})\right) dB^H(s_n) \tag{9}$$

The following result relates this iterated integral (9) and the multiple integral (8).

Theorem 3.1 *If $f \in L^2_{\phi,n}$, then the iterated integral (9) exists and*

$$I_n(f) = n! \int_{0\le s_1<\cdots<s_n\le t} f(s_1,\dots,s_n)dB^H(s_1)\cdots \times dB^H(s_n).$$

If $f \in L^2_{\phi,n}$ is a simple function of the form

$$f(t_1,\cdots,t_n) = \sum a_{i_1\cdots i_n} f_{i_1}(t_1)\cdots f_{i_n}(t_n),$$

then the ϕ-trace Tr_ϕ and its powers Tr^k_ϕ for $k \in \{1,2,\dots,[n/2]\}$ are defined as

$$\mathrm{Tr}^k_\phi f(t_1,\dots,t_{n-2k})$$
$$= \int_0^\infty\cdots\int_0^\infty f(s_1,\dots,s_{2k},t_1,\dots,t_{n-2k}) \times \phi(s_1-s_2)\cdots\phi(s_{2k-1}-s_{2k}) \times ds_1\cdots ds_{2k}.$$

To define the trace in general let γ_ε be an approximation to the Dirac function, that is,

$$\lim_{\varepsilon\downarrow 0}\int \gamma_\varepsilon(s,t)f(s)ds = f(t)$$

in some sense and

$$\int_0^\infty\int_0^\infty \gamma_\varepsilon(s,t)dsdt < \infty.$$

If $f \in L^2_{\phi,n}$, then $f^\varepsilon \in L^2_{\phi,n}$ where

$$f^\varepsilon(t_1,\cdots,t_n)$$
$$= \int_{\mathbb{R}^n_+} f(s_1,\dots,s_n)\gamma_\varepsilon(s_1,t_1)\cdots\gamma_\varepsilon(s_n,t_n)ds_1\cdots ds_n.$$

Let

$$\rho_\varepsilon(s,t) = \int_0^\infty \gamma_\varepsilon(s,u)\gamma_\varepsilon(t,u)du.$$

The kth ϕ-trace of f^ε is

$$\mathrm{Tr}^k_\phi f^\varepsilon(t_1,\dots,t_{n-2k})$$
$$= \int_{\mathbb{R}^n_+} f(s_1,\dots,s_n)\rho_\varepsilon(s_1,s_2)\cdots\rho_\varepsilon(s_{2k-1},s_{2k}) \times \gamma_\varepsilon(s_{2k-1},t_1)\cdots\gamma_\varepsilon(s_{2n},t_{n-2k}) \times ds_1\cdots ds_n.$$

The kth trace of f is said to exist if

$$\mathrm{Tr}^k_\phi f(t_1,\dots,t_{n-2k}) = \lim_{\varepsilon\to 0}\mathrm{Tr}^k_\phi f^\varepsilon(t_1,\dots,t_{n-2k}).$$

Now multiple Stratonovich integrals of a fractional Brownian motion are defined. Let

$$(B^H(t))^\varepsilon = \int_0^\infty \gamma_\varepsilon(t,s)dB^H(s)$$

and $f \in L^2_{\phi,n}$. Define

$$S^\varepsilon_n(f) = \int_{\mathbb{R}^n_+} f(s_1,\dots,s_n)(B^H(s_1))^\varepsilon\cdots(B^H(s_1))^\varepsilon \times ds_1\cdots ds_n. \tag{10}$$

If $S^\varepsilon_n(f)$ converges in $L^2(P)$ as $\varepsilon\to 0$, then the multiple Stratonovich integral is said to exist and is denoted

$$S_n(f) = \int_{\mathbb{R}^n_+} f(s_1,\dots,s_n)\delta B^H(s_1)\cdots\delta B^H(s_n). \tag{11}$$

It follows that

$$\left(\int_0^\infty f dB^H\right)^n = \sum_{k\le[n/2]}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}\left(\mathrm{Tr}^k_\phi f^{\otimes n}\right)$$

where $f^{\otimes n}$ is the symmetric tensor product of f. More generally, if $f_1,\dots,f_n \in L^2_\phi$ and $f \in L^2_{\phi,n}$ is the symmetrization of $f_1,\dots,f_n$, then

$$\int_0^\infty f_1 dB^H\cdots\int_0^\infty f_n dB^H$$
$$= \sum_{k\le[n/2]}\frac{n!}{2^k k!(n-2k)!}I_{n-2k}\left(\mathrm{Tr}^k_\phi(f)\right)$$

and $S^\varepsilon_n(f)$ can be defined as in (10). If for $k \in \{1,\dots,[n/2]\}$

$$\int_{\mathbb{R}^n_+} f(s_1,\dots,s_n)\gamma_\varepsilon(s_1,s_2)\cdots\gamma_\varepsilon(s_{2k-1},s_{2k}) \times \gamma_\varepsilon(s_{2k+1},t_1)\cdots\gamma_\varepsilon(s_{2n},t_{n-2k})ds_1\cdots ds_n.$$

converges to a function $\mathrm{Tr}_\phi^k f$ in $L^2_{\phi,n-2k}$ as $\varepsilon \to 0$ then $(S_n^\varepsilon(f),\, n \in \mathbb{N})$ converges in $L^2(P)$ and the limit, which is called the extended Hu-Meyer formula [6], is

$$S_n(f) = \sum_{k \le [n/2]} \frac{n!}{2^k k!(n-2k)!} I_{n-2k}\left(\mathrm{Tr}_\phi^k(f)\right).$$

For Brownian motion, there is a well known expansion of any square integrable functional on Wiener space in terms of multiple Wiener integrals [9] or Hermite polynomials. The following result is the analogue for a fractional Brownian motion with $H \in (1/2, 1)$.

Theorem 3.2 *If $F \in L^2(P)$, then there is a sequence $f_n \in L^2_{\phi,n}$, $n \in \mathbb{N}$) such that*

$$\sum_{n=1}^{\infty} |f_n|_\phi^2 < \infty$$

and

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} \int_{\mathbb{R}_+^n} f_n(s_1, \dots, s_n) \times dB^H(s_1) \cdots dB^H(s_n) \quad \text{a.s.} \tag{12}$$

The multiple integrals on the right hand side of (12) can be expressed as iterated integrals so that F can be expressed as a sum of a constant and a stochastic integral. This result has many applications in stochastic analysis.

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